Linear Algebra I
Exercise Sheet III: Properties of Linear Maps

Linearity

Determine whether the following maps are linear or not. If not, state why.

Question 1. $L : V \to V : v \mapsto 0$. (The trivial map)

Question 2. $L : V \to V : v \mapsto v$. (The identity map)
Related: explain why a change of coordinates is always given by the identity map.

Question 3. $L : \mathbb{R}^2 \to \mathbb{R} : (x, y, z) \mapsto x + z$.

Question 4. $L : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (xy + z, 3z)$.

Question 5. $L : \mathbb{R} \to \mathbb{R} : x \mapsto e^x$.

Question 6. $L : V_n \to V_n : f(x) \mapsto \frac{\mathrm{d}f}{\mathrm{dx}}$, where

$$V_n = \{a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \,|\, a_i \in \mathbb{Q}\}.$$

Question 7. $L : V_3 \to V_4 : f(x) \mapsto \int f \mathrm{dx}$.
Note: If $\frac{\mathrm{d}F}{\mathrm{dx}} = f(x)$ then $\int f(x)\mathrm{dx} = F(x) + C$, where $C \in \mathbb{R}$. This involves translation. How then can it be that this map is linear (which, in fact, it is)?

Question 8.

$$L : \{\text{differentiable functions from } \mathbb{R} \to \mathbb{R}\} \to \{\text{differentiable functions from } \mathbb{R} \to \mathbb{R}\} : f \mapsto \frac{\mathrm{d}f}{\mathrm{dx}}.$$

Question 9.

$$L : \{\text{differentiable functions from } \mathbb{R} \to \mathbb{R}\} \to \{\text{functions from } \mathbb{R} \to \mathbb{R}\} : f \mapsto \frac{\mathrm{d}f}{\mathrm{dx}}.$$

Question 10.

$$L : \{\text{differentiable functions from } \mathbb{R} \to \mathbb{R}\} \to \{\text{functions from } \mathbb{R} \to \mathbb{R}\} : f \mapsto \frac{\mathrm{d}^2 f}{\mathrm{dx}^2}.$$

Question 11.

$$L : \{\text{twice differentiable functions from } \mathbb{R} \to \mathbb{R}\} \to \{\text{functions from } \mathbb{R} \to \mathbb{R}\} : f \mapsto \frac{\mathrm{d}^2 f}{\mathrm{dx}^2}.$$

Trace, Determinant, and Rank

Recall that the trace of a diagonal matrix is the sum of diagonal entries.

Question 12. Find the trace, determinant, and rank of the following matrices.

$$A = \begin{pmatrix} 14 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -3 & -1 \\ 2 & -6 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & -2 & 2008 \\ 0 & 0 & -16 & 32 \\ 0 & 0 & 0 & \frac{3}{8} \end{pmatrix},$$

Question 13. The power of a matrix refers to products, so

$$A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_{n \text{ times}}.$$

Find the n-th power of the following matrices

$$X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Question 14. Consider the linear transformation represented by the square matrices A and B in different bases, and let P be the basis transformation matrix, so

$$PAP^{-1} = B.$$

What is B^{50} in terms of A?

Question 15. Recall that the determinant is a multiplicative function, that is,

$$\det(AB) = \det(A)\det(B).$$

Consider the linear map represented by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 8 & 3 & 10 \\ -8 & 0 & -7 \end{pmatrix}.$$

Find $\det(A^{263})$. Compare this with $\det(A)^{263}$.

<u>Kernel and Image</u>

Question 16. Find the kernel and image of the following maps. For each map verify the Rank-Nullity theorem.

- $L : \mathbb{R} \to \mathbb{R}^3 : x \mapsto (x, 3x, -2x)$.
- $L : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, 2y - z)$.
- $L : \mathbb{C}^3 \to \mathbb{C}^3 : (z, w, \omega) \mapsto (7z - w, 2\omega + w, 14z - 5w - 6\omega)$.
- $L : V_4 \to V_4 : f(x) \mapsto \frac{\mathrm{d}f}{\mathrm{dx}}$.
- $L : V_3 \to V_4 : f(x) \mapsto \int f(x)\mathrm{dx}$.

Question 17. Prove for any linear map $L : V \to W$ that $\mathrm{Im}(L)$ (image of L) and $\ker(L)$ (kernel of L) are vector spaces.

Question 18. What can you say about the image of a linear map from $\mathbb{R}^3 \to \mathbb{R}^3$ given by a 3×3 matrix with two real eigenvalues?